SYNOPSIS

Ph.D in Mathematics

OPERATION APPROACH OF δP_S-OPEN SETS IN TOPOLOGICAL SPACES, FUZZY TOPOLOGICAL SPACES AND NANO TOPOLOGICAL SPACES

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Introduction

Topology is a relatively new branch of mathematics; most of the research in topology has been done since 1900.Topology is the mathematical study of the properties that are preserved through deformations, twisting, and stretching of objects.Topologists freed themselves into a narrow approach to work in abstract spaces such as n-dimensional manifolds, projective spaces, Riemann surfaces, function space etc. Topology has developed as a field of study from geometry and set theory, through analysis of concepts such as space, dimension and transformation and emerged with the rudimentary knowledge of Set Theory, Algebra, Geometry and Analysis. The purpose of the topology is to elucidate and investigate the ideas of continuity and connectivity within the frame work of mathematics. The study of topological spaces and general properties makes up one branch of topology known as 'general topology'. General Topology has exhibited its elegance in both pure and applied aspects so that one can visualize the existence of topological structures and the usage of topological properties in various application fields.

Levine first proposed the idea of semi-open sets in 1963 [10]. Velicko made the first note on the class of δ-open subsets of a topological space in 1968 [20]. The research of various topological space properties relies heavily on this class of sets. As that day, numerous authors have defined new classes of sets in topological spaces using this class. Mashhour, Abd El-Monsef, and El-Deeb presented the concept of preopen sets in 1982 [11]. Raychaudhuri and Mukherjee [14] initially introduced and explored a class of sets known as δ-preopen. On the contrary, Kasahara introduced the concept of α-closed graphs of functions and explored the approach of an operation on τ in 1979 [9]. Ogata had renamed the operation α to operation γ operation on τ in 1991[13]. Khalaf and Asaad in 2009 [7] introduced a new concept called P_S-open sets in topological spaces. A preopen subset P of a space W is called as P_S -open set if for all $p \in P$ there exist a semiclosed set F such that $p \in F \subseteq P$. Assad introduced operation approaches on P_S-open sets and its separation axioms in 2016 [2]. Vidhyapriya et al in 2020 [21] defined a new kind of open sets in topological spaces (W, τ) called δP_S-open sets combining the concepts of δ-preopen and semi closed sets. This class of sets lies between the classes δ-preopen and P_S-open sets.

In this thesis, the author has introduced the concept of an operation ϱ on the class of δP_S-open subsets in topological spaces as a mapping $\varrho: \delta P_S O(W) \rightarrow \mathcal{P}(W)$ and studied the concepts of ϱ-open sets.

Objectives and Scope of the research

- To acquire knowledge and understanding of the concepts in reviewing the work of various authors.
- To define a new type of operation on topological spaces using δP_S-open sets.
- To obtain the properties of newly introduced concept.
- To enhance the characterizations.
- To analyze the new concept with other existing concepts and infer the differences.
- To apply the new concept to analyse a situation as a step to applied research.

Methodology

The research methods pertained in the present study are as follows

- Analytical method of comparing ϱ-open sets with existing open sets.
- Interpreting the results as diagrams.
- Producing counter examples wherever necessary to substantiate the result.
- Analysis of preservation of topological properties by ϱ-open sets.
- Obtaining characterisation theorems.
- Apply the developed theory to suitable real life scenario and test its efficiency.

Notions

Throughout the thesis, the following notations are used

- (W, τ), (Z, σ) and (Y, η) denote non-empty topological spaces on which no separation axioms are mentioned, unless it is stated specifically.
- In all the diagrams, $A \rightarrow B$ represents A implies B but not conversely and $A \nleftrightarrow B$ represents A and B are independent of each other.

Chapter 1

Chapter 1 deals with the study of the preliminary definitions and results which are used to accomplish the research work.

Chapter 2

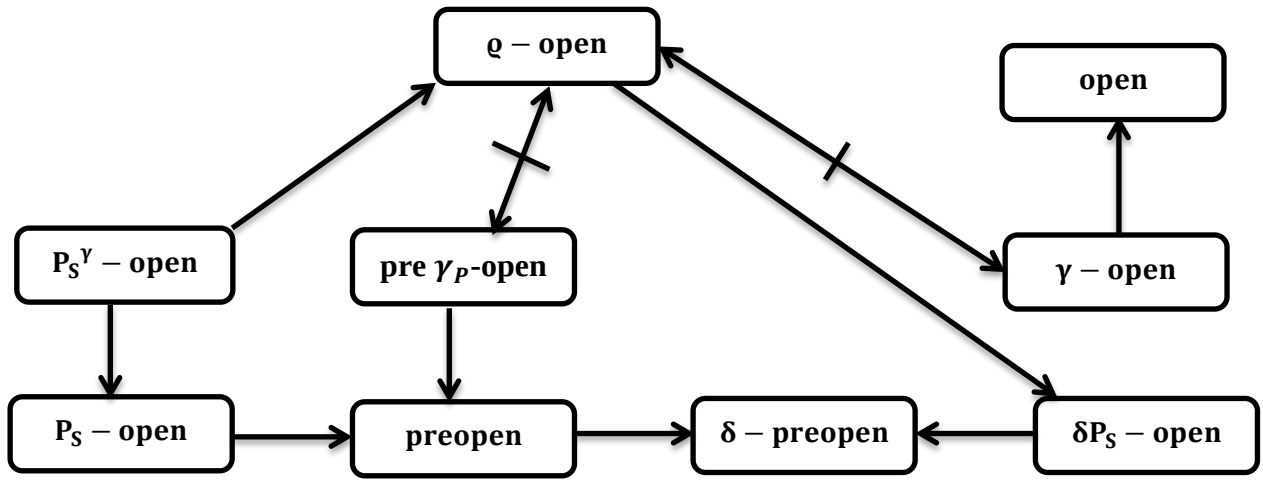
In this chapter, the author has introduced the concept of an operation ϱ on δP_S-open subsets in topological spaces as a mapping $\varrho: \delta P_S O(W) \to \mathcal{P}(W)$ such that $H \subseteq H^{\varrho}$ for every $H \in \delta P_S O(W)$. Then ϱ-open sets, ϱ-regular spaces, $closure_{\varrho}$ and ϱ-closure of subsets are analysed with their properties.

ϱ-open set. Let (W, τ) be a topological space and $\varrho : \delta P_S O(W) \to \mathcal{P}(W)$ be an operation on $\delta P_S O(W)$. A nonempty set P of W is said to be a ϱ-open set if for each $p \in P$, there exist a δP_S-open set H such that $p \in H$ and $H^{\varrho} \subseteq P$.

The properties of ϱ-open sets analysed in this chapter are given below.

- Every ϱ-open set is a δP_S-open set.
- γ-open sets and ϱ-open sets are independent.
- Every ${P_S}^{\gamma}$-open set is a ϱ-open set.
- Pre γ_P-open sets and ϱ-open sets are independent.
- In a locally indiscrete space(W, τ),the class of ϱ-open sets coincides with the class of γ-open sets, if the operations ϱ and γ are the same functions.
- In a s-regular space, every γ-open set is ϱ-open, if the operations ϱ and γ are the same functions.
- In a hyper connected space (W, τ), if an operation ϱ is defined on $\delta P_S O(W)$ then $\varrho O(W) = \{W, \emptyset\}$.
- In an indiscrete topological space(W, τ), $\varrho O(W) = \{W, \emptyset\}$.

The following diagram shows the relation of ϱ-open sets with other open sets in topological spaces:



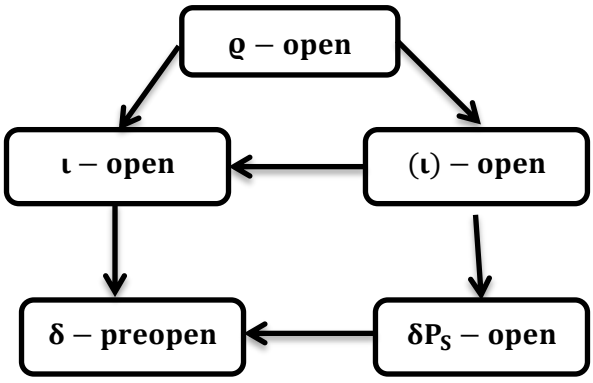
Chapter 3

The aim of this chapter is to introduce some topological properties such as limit points, derived sets, neighbourhood, interior, kernel, exterior, boundary, frontier and saturated sets based on ϱ-operation and their behaviour are discussed. Moreover, the author had introduced grill ϱ-space and discussed its properties. A new topology associated with this ϱ- grill is obtained.

Chapter 4

Cuong and Maki in 2008 [1] defined and investigated the concept of the mapping γ_P on the collection of all preopen subsets of (W, τ) and introduced the notion of pre γ_P-open sets and studied some of their properties. Khalaf and Ibrahim proposed operation approaches via preopen sets and its continuous mappings in 2012 [8] in a different way.

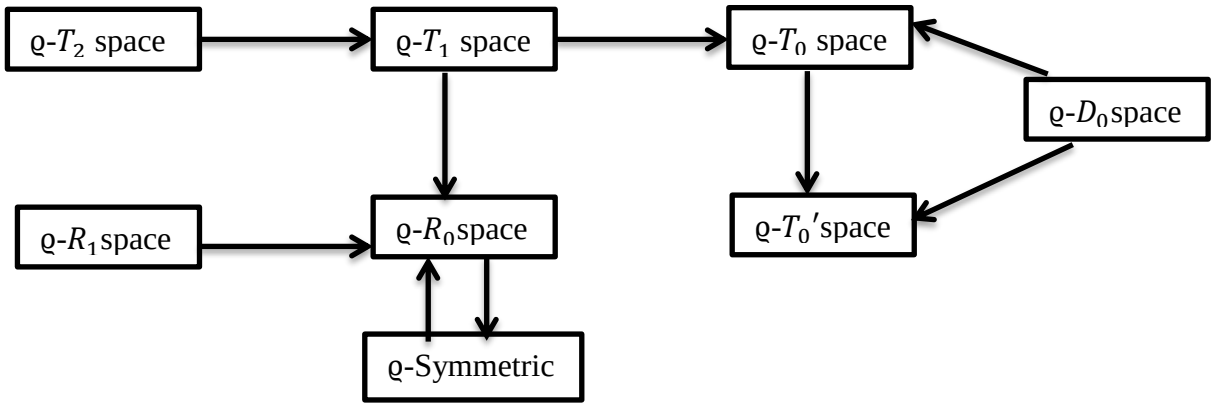
In a similar way, in this thesis, the author has introduced two types of open sets ι-open sets and (ι)-open sets on the class of δ-preopen subsets in topological spaces. Then the corresponding two types of closures and their properties are analysed. The comparison between ι-open sets and (ι)-open sets with ϱ-open sets are brought out and listed as below.



Chapter 5

In this Chapter, the author has introduced various spaces using ϱ-open sets, such as ϱ-T_n and ϱ-T'_n for the values of n = 0,1,2 and their properties and relation with other spaces are studied. The concepts of ϱ-generalized closed sets, ϱ-$T_{1/2}$ spaces and symmetric spaces on ϱ-operation are introduced and their relations among them are studied. Some derived sets, ϱ-R_0 and ϱ-R_1 spaces are discussed with their properties.

The relationship between ϱ-spaces are analysed below.



Chapter 6

In this Chapter, the author has introduced the concept of δP_S-$(\alpha_\varrho, \beta_\varrho)$-continuity and analysed its properties. Another mapping on ϱ-open sets called $(\alpha_\varrho, \beta_\varrho)$-continuous mapping is also introduced and its properties are discussed. The different types are given below.

Let $\alpha_\varrho: \delta P_S O(W, \tau) \to \mathcal{P}(W)$ and $\beta_\varrho: \delta P_S O(Z, \sigma) \to \mathcal{P}(Z)$ be two ϱ-operations on $\delta P_S O(W)$ and $\delta P_S O(Z)$ respectively.

δP_S-$(\alpha_\varrho, \beta_\varrho)$-continuous mappings. A mapping $f: (W, \tau) \to (Z, \sigma)$ is said to be a $\delta P_S(\alpha_\varrho, \beta_\varrho)$-continuous mapping if for every $p \in W$ and each δP_S-open set V in (Z, σ) containing $f(p)$ there exists a δP_S-open set U in (W, τ) such that $p \in U$ and $f(U^{\alpha_\varrho}) \subseteq V^{\beta_\varrho}$.

$(\alpha_\varrho, \beta_\varrho)$-continuous mappings. A mapping $f: (W, \tau) \to (Z, \sigma)$ is said to be a $(\alpha_\varrho, \beta_\varrho)$-continuous if for every $p \in W$ and each β_ϱ-open set V in (Z, σ) containing $f(p)$ there exists a α_ϱ-open set U in (W, τ) such that $p \in U$ and $f(U) \subseteq V$.

In general,

δP_S-$(\alpha_\varrho, \beta_\varrho)$- continuity ⟶ **$(\alpha_\varrho, \beta_\varrho)$-continuity**

Somewhat continuity of δP_S-$(\alpha_\varrho, \beta_\varrho)$-operation on topological spaces is introduced and studied.

Chapter 7

For the same topological space two different operations γ and γ' are defined and a new concept of bioperation is introduced by Umeshara, Maki and Nori [19] in the year 1992 using union and intersection of γ and γ' values. This chapter deals with the concepts of bioperation using intersection of two operations ϱ and ϱ' in a topological space .A

topological space (W,τ) equipped with two operations namely ϱ and ϱ'defined on the class δP_S-open sets is called **bioperation $\boldsymbol{\varrho}$-topological space** and it is denoted by $(\boldsymbol{W},\boldsymbol{\tau},\boldsymbol{\varrho},\boldsymbol{\varrho'})$**.** Then, as an extension, $[\varrho,\varrho']$-open sets, $[\varrho,\varrho']$-regular spaces, two types of $[\varrho,\varrho']$-closures are introduced and their properties are studied.

$[\boldsymbol{\varrho},\boldsymbol{\varrho'}]$**-open.** Let ϱ and ϱ' be two operations on the collection of δP_S-open sets of (W,τ). A non-empty subset P of W is said to be $[\varrho,\varrho']$-open if for all $p\in P$ there exists a pair of δP_S-open sets $U$and V having p such that $U^{\varrho}\cap V^{\varrho'}\subseteq P$. The collection of all $[\varrho,\varrho']$-open sets is denoted by $[\varrho,\varrho']O(W)$.

Submaximal spaces are analysed using $[\varrho,\varrho']$-open sets. The concept of bioperation continuity is also discussed .Weakly continuous functions and Somewhat Continuous functions are established.

$[\boldsymbol{\varrho},\boldsymbol{\varrho'}]$**-submaximal space.** A bioperation topological space $(W,\tau,\varrho,\varrho')$ is said to be a $[\varrho,\varrho']$-submaximal space if every $[\varrho,\varrho']$-dense subset of W is $[\varrho,\varrho']$-open.

$[\boldsymbol{\alpha_{\varrho}},\boldsymbol{\alpha_{\varrho'}}]$-$[\boldsymbol{\beta_{\varrho}},\boldsymbol{\beta_{\varrho'}}]$-$\boldsymbol{\delta P_S}$**-continuous mappings.** A mapping $f:(W,\tau,\alpha_{\varrho},\alpha_{\varrho'})\to(Z,\sigma,\beta_{\varrho},\beta_{\varrho'})$ is said to be $[\alpha_{\varrho},\alpha_{\varrho'}]$-$[\beta_{\varrho},\beta_{\varrho'}]$-$\delta P_S$-continuous if for each $p\in W$ and each pair of δP_S-open sets S and T containing $f(p)$ there exist δP_S-open sets U and V containing p such that $f(U^{\alpha_{\varrho}}\cap V^{\alpha_{\varrho'}})\subseteq S^{\beta_{\varrho}}\cap T^{\beta_{\varrho'}}$.

Weakly $[\boldsymbol{\alpha_{\varrho}},\boldsymbol{\alpha_{\varrho'}}]$-$[\boldsymbol{\beta_{\varrho}},\boldsymbol{\beta_{\varrho'}}]$-$\boldsymbol{\delta P_S}$**-continuous mappings.** A mapping $f:(W,\tau,\alpha_{\varrho},\alpha_{\varrho'})\to(Z,\sigma,\beta_{\varrho},\beta_{\varrho'})$ is said to be a Weakly $[\alpha_{\varrho},\alpha_{\varrho'}]$-$[\beta_{\varrho},\beta_{\varrho'}]$-$\delta P_S$-continuous mapping if for all $p\in W$ and for ever $[\beta_{\varrho},\beta_{\varrho'}]$-open set O of Z containing $f(p)$ there exists a δP_S-open set P of W containg p such that $f(P)\subseteq[\beta_{\varrho},\beta_{\varrho'}]Cl(O)$.

Somewhat $[\boldsymbol{\alpha_{\varrho}},\boldsymbol{\alpha_{\varrho'}}][\boldsymbol{\beta_{\varrho}},\boldsymbol{\beta_{\varrho'}}]$**-continuous mappings.** A mapping $f:(W,\tau,\alpha_{\varrho},\alpha_{\varrho'})\to(Z,\sigma,\beta_{\varrho},\beta_{\varrho'})$ is said to be somewhat $[\alpha_{\varrho},\alpha_{\varrho'}]$-$[\beta_{\varrho},\beta_{\varrho'}]$-$\delta P_S$-continuous if for every $[\beta_{\varrho},\beta_{\varrho'}]$-open set O of Z and $f^{-1}(O)\neq\emptyset$, there exists a δP_S-open set P in W such that $P\neq\emptyset$, $P\subseteq f^{-1}(O)$.

Chapter 8

The development of crisp topology originated from classical analysis and it has numerous applications in many research areas like data mining, machine learning, quantum gravity and data analysis. The notion of topology discloses the relationship between spatial elements and

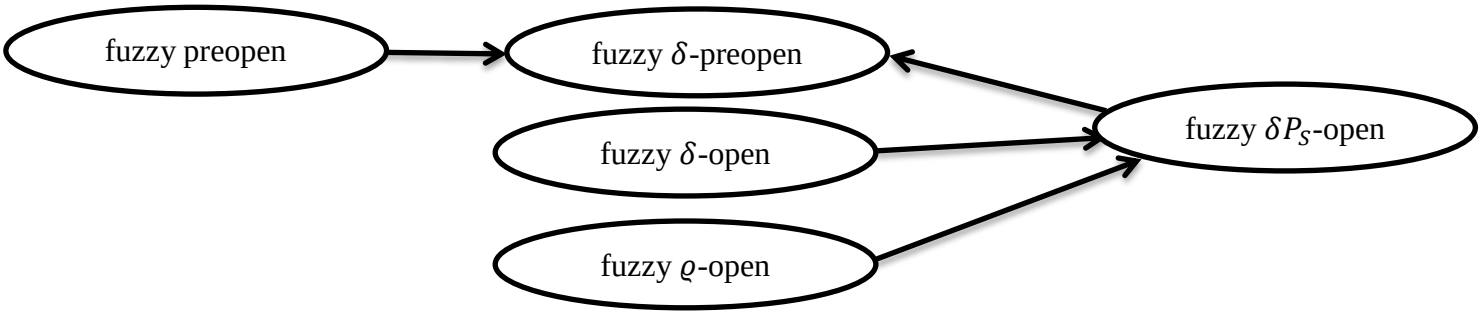
features. In recent years topology has not firmly established itself as an important part of general mathematical structure but also of physicists mathematical arsenal. Zadeh [22] in 1965 generalized the usual set by using fuzzy set in which every detail is described with a degree of membership function. The fuzzy set has many programs in economic system, decision making, facts mining, commercial enterprise and many other. The notion of fuzzy topological space is introduced by Chang [6] in 1968. He generalized some fundamental concepts of topology such as open sets, closed sets, neighbourhood and continuity. Thivagar et al [18] in 2003 introduced the concept of nano topology which is defned in terms of the lower and upper approximations and the boundary region of a subset of an universe. Also established the weak forms of nano open sets namely nano α-open sets, nano pre-open sets and nano semi-open sets in a nano topological space and also they introduced nano-regular open sets. Senapati and Yager [17] in 2020 introduced Fermatean Fuzzy Sets in which the sum of the cubes of membership and non-membership is bounded by zero and one. This chapter deals with ϱ-open sets in various other topological spaces such as ϱ-open set in fuzzy topological spaces and analysed some of its properties, maximal and minimal open sets, Fermatean ϱ fuzzy sets and an edge detection is made using this set. Also, ϱ-open set in nano topological spaces is discussed along with an illustration using attributes and objects.

Fuzzy ϱ-open set. Let (W, τ) be a fuzzy topological space and $f_\varrho: f\delta P_S O(W) \to \mathcal{P}(W)$ be an operation on the collection of $f\delta P_S O(W)$. A nonempty fuzzy set λ of W is said to be a fuzzy ϱ-open set if for each $x \in \lambda$ there exists a fuzzy δP_S-open set ζ such that $x \in \zeta$ and $\zeta^{f_\varrho} \leq \lambda$ where ζ^{f_ϱ} is the image of ζ under f_ϱ and always $\zeta \subseteq \zeta^{f_\varrho}$.The family of all fuzzy ϱ-open set is denoted by $f\varrho O(W)$.

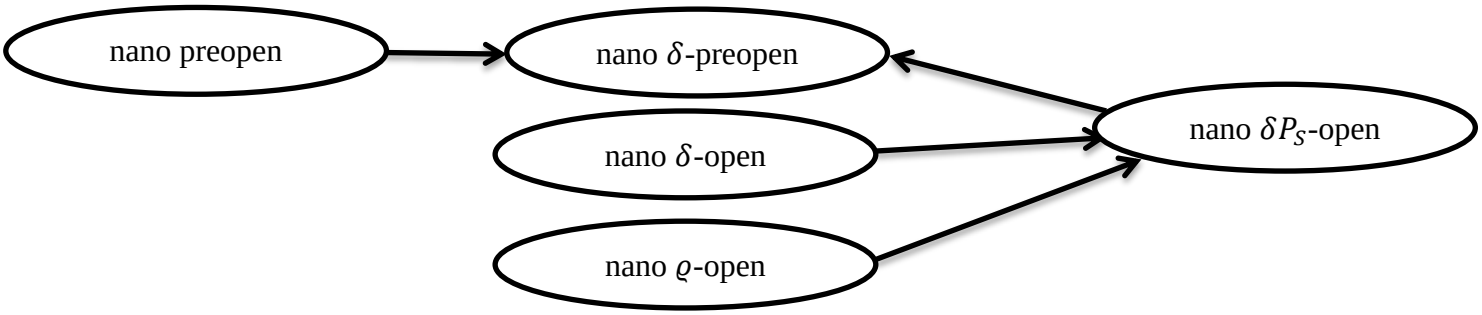
Fermatean fuzzy ϱ open set. Let (W, τ_f) be a fuzzy topological space and $f_\varrho: f\delta P_S O(W) \to I^W$ be an operation on the collection of $f\delta P_S O(W)$. A Fermatean fuzzy ϱ open set ς is defined as $\varsigma = \{(x, \zeta(x), (1-\zeta)(x));\ x \in W \text{ and } \zeta^{f_\varrho} \subseteq \varsigma\}$ where $\zeta(x), (1-\zeta)(x)$ denote the grades of membership and non-membership respectively with a constraint $0 \leq \zeta^3(x) + (1-\zeta)^3(x) \leq 1$.

Nano ϱ-open set. Let $(U, \tau_R(W))$ be a nano topological space and $N_\varrho: N\delta P_S O(U, W) \to \mathcal{P}(U)$ be a nano ϱ-operation on the collection of $N\delta P_S O(U, W)$. A nonempty set K of U is said to be a nano ϱ-open set if for each $p \in K$ there exists a nano δP_S-open set L such that $p \in L$ and $L^{N_\varrho} \subseteq K$ where L^{N_ϱ} is the image of L under N_ϱ and always $L \subseteq L^{N_\varrho}$. The family of all nano ϱ-open sets is denoted by $N\varrho O(U, W)$.

The relationship between fuzzy ϱ-open set with other fuzzy open sets



The relationship between nano ϱ-open set with other nano open sets



Conclusion

Thus in this thesis, the author has introduced a new type of operation approach and studied its corresponding open sets, closed sets, topological properties, separation axioms, continuous mappings and bioperation and also extended to fuzzy topological spaces and nano topological spaces. Their properties are analyzed and many characterization theorems with necessary counter examples are provided. Another kind of operation approach ι is introduced and discussed its corresponding open sets, closed sets, topological properties, separation axioms. Defined Fermatean fuzzy ϱ-open sets and based on this edge detection is made. Using nano ϱ-open sets the main cores are obtained that to be noticed while selecting a car model from Maruti Suzuki brands.

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